

# $^1S_0$ nucleon-nucleon scattering in the modified Weinberg approach

E. Epelbaum,<sup>1</sup> A. M. Gasparyan,<sup>1,2</sup> J. Gegelia,<sup>1,3</sup> and H. Krebs<sup>1</sup>

*<sup>1</sup>Institut für Theoretische Physik II,  
Fakultät für Physik und Astronomie,*

*Ruhr-Universität Bochum 44780 Bochum, Germany*

*<sup>2</sup>SSC RF ITEP, Bolshaya Cheremushkinskaya 25, 117218 Moscow, Russia*

*<sup>3</sup>Tbilisi State University, 0186 Tbilisi, Georgia*

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## Abstract

Nucleon-nucleon scattering in the  $^1S_0$  partial wave is considered in chiral effective field theory within the renormalizable formulation of Ref. [1] beyond the leading-order approximation. By applying subtractive renormalization, the subleading contact interaction in this channel is taken into account non-perturbatively. For a proper choice of renormalization conditions, the predicted energy dependence of the phase shift and the coefficients in the effective range expansion are found to be in a good agreement with the results of the Nijmegen partial wave analysis.

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## I. INTRODUCTION

The seminal work of Weinberg [2] has triggered a renewed interest to the nuclear force problem in the framework of effective field theory (EFT). In this approach, nuclear forces are defined as kernels of the corresponding dynamical equations and can be derived order-by-order making use of the systematic chiral expansion.

Starting from the pioneering work of Ref. [3], this approach has developed rapidly and is nowadays commonly employed in studies of low-energy few- and many-nucleon dynamics and nuclear structure calculations, see [4–6] for recent review articles. While offering many attractive features, Weinberg’s approach was criticized for being non-renormalizable. The main difficulty is related to the fact that iterations of the truncated NN potential within the Lippmann-Schwinger (LS) equation generate ultraviolet (UV) divergencies which cannot be absorbed by counter terms (contact interactions) included in the truncated potential. In particular, infinitely many counter terms are needed to absorb UV divergences emerging from iterations of the leading-order (LO) one-pion exchange (OPE) potential [7]. This feature is sometimes referred to as inconsistency of Weinberg’s approach.

The UV cutoff  $\Lambda$  can be removed from the LS equation by enforcing the limit  $\Lambda \rightarrow \infty$  non-perturbatively, see e.g. [8, 9]. It is possible to obtain a finite, manifestly non-perturbative solution of the LS equation with a singular  $1/r^3$ -potential by including one/no contact operator in each attractive/repulsive channel [9]. However, such a procedure is incompatible with the principles of EFT which require that all UV divergences emerging from iterations of the LS equation are absorbed by counter terms [10]. It is not surprising that such an approach fails to reproduce experimental data even at N<sup>3</sup>LO [11].

Treating the exchange of pions perturbatively as suggested by Kaplan, Savage and Wise (KSW) [12] allows one to avoid the above-mentioned inconsistency. However, the perturbative series fail to converge within this framework [13–16].

Presently, there exist different views and formulations of organizing the chiral expansion in the few-nucleon sector [8–10, 17–32]. A novel approach to the NN scattering problem in EFT has been formulated in Refs. [1, 33–36] and is referred to as the modified Weinberg approach. Within this framework, the leading order (LO) NN scattering amplitude is obtained by solving the Kadyshevsky equation [37] for the LO potential consisting of the contact interaction part and the OPE potential. This equation provides an example of three-dimensional integral equations which satisfy relativistic elastic unitarity. An important feature of the Kadyshevsky equation is that it is renormalizable for the LO potential, i.e. all ultraviolet divergences generated by iterations can be explicitly absorbed into redefinition of the NN derivative-less contact interaction. The scattering amplitude can still be renormalized if higher-order corrections to the potential are taken into account perturbatively. If higher order corrections to the potential indeed provide small contributions to the amplitude, their perturbative and non-perturbative inclusions are expected to lead to small differences in the results which are beyond the accuracy one is working at. However, this observation is only meaningful if a proper renormalization is carried out in both cases. In general, we are not able to subtract all divergences from amplitudes if higher-order contributions in the potential are treated non-perturbatively. In the  $^1S_0$  partial wave, one observes a very large discrepancy between the LO EFT results and the experimental data already at rather low energies [1]. This large discrepancy signals that at least a part of the higher-order contributions in the effective potential is likely to require a non-perturbative treatment within our

approach.<sup>1</sup>

In this paper we study in detail the role of the next-to-leading order (NLO) short-range contribution to the potential which can be included both perturbatively and non-perturbatively. Specifically, we will express the solution to the integral equation in a closed form following the lines of Ref. [40] and apply the BPHZ-type subtractive renormalization [41]. After subtracting *all* ultraviolet divergences, we will calculate the remaining finite expressions numerically, fit the available two low-energy constants (LECs) to the data and compare the obtained results with the phase shifts for various choices of the renormalization scale parameter. Here and in what follows, the resulting NN amplitude will be referred to as NLO as opposed to the LO result of Ref. [1]. A more complete calculation including the corresponding two-pion exchange potential to first order in perturbation theory is postponed for a future study.

Our paper is organized as follows: In section II we provide the formal expression for the scattering amplitude by making use of the standard two-potential formalism. Subtractive renormalization of the amplitude is discussed in detail in section III. Next, section IV addresses the issue of the appropriate choice of the renormalization conditions (i.e. subtraction scale) and also presents the results of our calculation. Our findings are summarized in section V.

## II. FORMAL EXPRESSION FOR THE SCATTERING AMPLITUDE

In the framework of the modified Weinberg approach, the NLO  $^1S_0$  partial wave NN scattering amplitude<sup>2</sup> can be obtained by extracting the  $S$ -wave component from the solution to the integral equation (for the fully off-shell amplitude  $T$ )

$$T(p_0, \vec{p}', \vec{p}) = V(\vec{p}', \vec{p}) + \int d^3k V(\vec{p}', \vec{k}) G(p_0, k) T(p_0, \vec{k}, \vec{p}), \quad (2.1)$$

$$G(p_0, k) = \frac{m^2}{2(2\pi)^3} \frac{1}{(\vec{k}^2 + m^2) (p_0 - \sqrt{\vec{k}^2 + m^2} + i\epsilon)}, \quad (2.2)$$

where  $\vec{p}$  ( $\vec{p}'$ ) is the incoming (outgoing) three-momentum of the nucleon in the center-of-mass frame,  $p_0 = \sqrt{\vec{q}^2 + m^2}$  with  $m$  denoting the nucleon mass and  $\vec{q}$  being the corresponding three-momentum of an incoming (on-mass-shell) nucleon. Further, the potential is given by

$$\begin{aligned} V(\vec{p}', \vec{p}) &= [C + C_2 (\vec{p}'^2 + \vec{p}^2)] - \frac{g_A^2 M_\pi^2}{4F_\pi^2} \frac{1}{(\vec{p}' - \vec{p})^2 + M_\pi^2} \\ &\equiv V_C + V_\pi, \\ C &= C_S - 3C_T + \frac{g_A^2}{4F_\pi^2} + D M_\pi^2. \end{aligned} \quad (2.3)$$

Here  $g_A$ ,  $F_\pi$  and  $M_\pi$  are the nucleon axial-vector coupling, pion decay constant and the pion mass, respectively. The parameters  $C_S$ ,  $C_T$ ,  $C_2$  and  $D$  refer to the LECs of the effective

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<sup>1</sup> Notice, however, that the LO calculations reported in Refs. [38, 39] within the standard nonrelativistic approach using a coordinate-space regularization for the OPE potential yields a superior description of the phase shift.

<sup>2</sup> Note the different overall sign in comparison with the Feynman amplitude considered in Ref. [1].

Lagrangian. Below, we work with the  $S$ -wave component of Eq. (2.1) and denote the  $^1S_0$  partial wave projected OPE potential by  $V_\pi(p', p)$  with

$$V_\pi(p', p) = \frac{g_A^2 M_\pi^2}{16 F_\pi^2 p p'} \ln \frac{(p - p')^2 + M_\pi^2}{(p + p')^2 + M_\pi^2}. \quad (2.4)$$

For the analysis of divergent integrals, it is useful to have the asymptotics of  $V_\pi(p', p)$  at large values of momenta

$$V_\pi(p', p) \Big|_{p \rightarrow \infty, p' < \infty} \approx -\frac{g_A^2 M_\pi^2}{4 F_\pi^2 p^2}, \quad V_\pi(p + l, p) \Big|_{p \rightarrow \infty, |l| < \infty} \approx -\frac{g_A^2 M_\pi^2}{8 F_\pi^2 p^2} \ln p. \quad (2.5)$$

The contact-interaction part of the potential  $V_C$  is separable. Therefore, it is possible to write the solution to Eq. (2.1) in a form, which allows one to carry out the subtractive renormalization explicitly. This can be achieved by making use of the well-known two-potential formalism. For this purpose, we write Eq. (2.1) symbolically as

$$T = V + V G T, \quad (2.6)$$

and express its solution as

$$T = T_\pi + (1 + T_\pi G) T_C (1 + G T_\pi), \quad (2.7)$$

where  $T_\pi$  and  $T_C$  satisfy the equations

$$T_\pi = V_\pi + V_\pi G T_\pi, \quad (2.8)$$

$$T_C = V_C + V_C G (1 + T_\pi G) T_C. \quad (2.9)$$

For a separable contact-interaction potential,

$$V_C(p', p) = \xi(p')^T \mathcal{C} \xi(p), \quad (2.10)$$

where  $\mathcal{C}$  and  $\xi(p)$  are  $2 \times 2$  and  $2 \times 1$  matrices, respectively, whose explicit form will be specified below, the solution to Eq. (2.9) is also given in a separable form

$$T_C(p_0, p', p) = \xi^T(p') \mathcal{X} \xi(p), \quad (2.11)$$

where  $\mathcal{X}$  is a  $2 \times 2$  matrix,

$$\mathcal{X} = [\mathcal{C}^{-1} - \Sigma]^{-1}, \quad (2.12)$$

and the  $2 \times 2$  “selfenergy” matrix  $\Sigma$  reads

$$\begin{aligned} \Sigma(p_0) &= \xi G \xi^T + \xi G T_\pi G \xi^T \\ &\equiv \int d^3 k \xi(k) G(p_0, k) \xi^T(k) + \int d^3 k_1 d^3 k_2 \xi(k_1) G(p_0, k_1) T_\pi(p_0, k_1, k_2) G(p_0, k_2) \xi^T(k_2). \end{aligned} \quad (2.13)$$

Thus, the final expression of the amplitude  $T$  has the form

$$T = T_\pi + \Xi^T \mathcal{X} \Xi. \quad (2.14)$$

with

$$\Xi(p_0, p) = \xi(1 + G T_\pi) \equiv \xi(p) + \int d^3 k \xi(k) G(p_0, k) T_\pi(p_0, k, p). \quad (2.15)$$

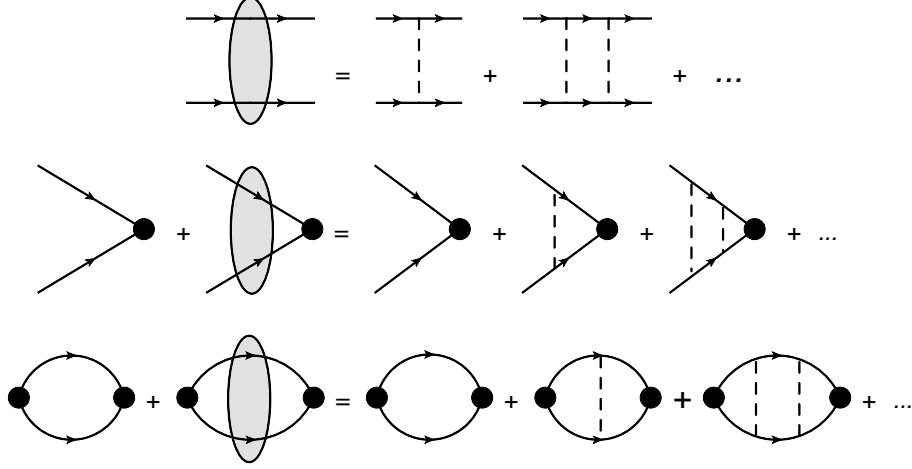


FIG. 1: Building blocks of the scattering amplitude. The first, second and third lines represent  $T_\pi$ ,  $\Xi$  and  $\Sigma$ , respectively. The solid and dashed lines correspond to nucleons and pions, respectively. The filled circles represent  $\xi$  and  $\xi^T$ .

### III. RENORMALIZATION OF THE SCATTERING AMPLITUDE

The expression for the scattering amplitude in Eq. (2.14) contains UV divergences. We perform renormalization by applying the BPHZ procedure, i.e. we subtract all divergences and sub-divergences of the loop diagrams and replace the LECs with their renormalized, finite values. In general, in renormalizable theories, subtractive renormalization can be realized by counter terms in the Lagrangian. Chiral effective field theory is renormalizable in the sense of effective field theories, i.e. all divergences can be absorbed into redefinition of an infinite number of counter terms. To realize subtractive renormalization in the considered problem, we would need to include the contributions of an infinite number of counter terms of the effective Lagrangian. Although this is possible for the case at hand by considering energy-dependent counter terms, here we only show explicitly one momentum- and energy-independent counter term  $\delta z$  and write the contact interaction potential in a separable form

$$V_C(p', p) = C + C_2 (p'^2 + p^2) = (1, p'^2 + \delta z) \begin{pmatrix} \tilde{C} & C_2 \\ C_2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ p^2 + \delta z \end{pmatrix}. \quad (3.1)$$

The new parameter is expressed as

$$\tilde{C} = C - 2 C_2 \delta z. \quad (3.2)$$

Thus, the contact-interaction potential has the form

$$\mathcal{C} = \begin{pmatrix} \tilde{C} & C_2 \\ C_2 & 0 \end{pmatrix}, \quad \xi(p) \equiv (\xi_1(p), \xi_2(p))^T = (1, p^2 + \delta z)^T. \quad (3.3)$$

The various terms contributing to the amplitude  $T$  are visualized diagrammatically in Fig. 1 in terms of the corresponding building blocks, where in the first line the amplitude  $T_\pi$  is shown. The second line represents  $\Xi$ , while the analogous diagrams for  $\Xi^T$  are not shown explicitly. The third line depicts the quantity  $\Sigma$  which contributes to  $\mathcal{X}$ , see Eq. (2.12). To obtain the amplitude using Eq. (2.14), we first perform subtractive renormalization and

afterwards calculate numerically the remaining finite expressions for the quantities  $T_\pi$ ,  $\Xi$ , and  $\mathcal{X}$ . In the following, we describe in detail how these quantities are renormalized. Since the amplitude  $T_\pi$  is finite (the ultraviolet regularity of the equation for  $T_\pi$  (2.8) follows from the asymptotics (2.5)), we begin our discussion with the subtractive renormalization of  $\Xi$ . By writing  $\Xi(p_0, p)$  as a perturbative series as shown in Fig. 1,

$$\Xi = \xi + \xi G V_\pi + \xi G V_\pi G V_\pi + \cdots, \quad (3.4)$$

it is easily seen that  $\Xi(p_0, p) = (\Xi_1(p_0, p), \Xi_2(p_0, p))^T$  can be obtained by solving the integral equation

$$\Xi = \xi + \Xi G V_\pi. \quad (3.5)$$

This expression defines a system of equations for the quantities  $\Xi_{1,2}(p_0, p)$ , which, using the explicit form of  $\xi(p)$  from Eq. (3.3), can be written as

$$\Xi_1(p_0, p) = 1 + \int d^3k \Xi_1(k) G(p_0, k) V_\pi(k, p), \quad (3.6)$$

$$\Xi_2(p_0, p) = p^2 + \delta z + \int d^3k \Xi_2(k) G(p_0, k) V_\pi(k, p). \quad (3.7)$$

The equation for  $\Xi_1(p_0, p)$  is free of ultraviolet divergences, see Eq. (2.5), and has an ultraviolet behavior  $\Xi_1(p_0, p) \xrightarrow{p \rightarrow \infty} \text{const.}$  On the other hand, to identify the divergences in  $\Xi_2(p_0, p)$ , it is convenient to consider iterations of Eq. (3.7)

$$\begin{aligned} \Xi_2(p) &= p^2 + \delta z + \int d^3k (k^2 + \delta z) G(p_0, k) V_\pi(k, p) \\ &+ \int d^3k d^3l (k^2 + \delta z) G(p_0, k) V_\pi(k, l) G(p_0, l) V_\pi(l, p) + \cdots \end{aligned} \quad (3.8)$$

Remembering the definition of  $G(p_0, k)$  in Eq. (2.2), we simplify

$$\begin{aligned} k^2 G(p_0, k) &= q^2 G(p_0, k) - \frac{m^2}{2(2\pi)^3} \frac{p_0 + \sqrt{\vec{k}^2 + m^2}}{\vec{k}^2 + m^2} \\ &\equiv q^2 G(p_0, k) + \tilde{G}(p_0, k). \end{aligned} \quad (3.9)$$

Substituting the above expression into Eq. (3.8) and re-organizing the perturbative series we obtain

$$\begin{aligned} \Xi_2(p_0, p) &= p^2 + q^2 \int d^3k G(p_0, k) V_\pi(k, p) \\ &+ q^2 \int d^3k d^3l G(p_0, k) V_\pi(k, l) G(p_0, l) V_\pi(l, p) + \cdots \\ &+ \delta z + \int d^3k \tilde{G}(p_0, k) V_\pi(k, p) \\ &+ \int d^3k d^3l \left[ \delta z + \tilde{G}(p_0, k) V_\pi(k, l) \right] G(p_0, l) V_\pi(l, p) + \cdots \end{aligned} \quad (3.10)$$

From this equation it is easily seen that  $\Xi_2(p_0, p)$  can be written in the form

$$\Xi_2(p_0, p) = p^2 + q^2 [\Xi_1(p_0, p) - 1] + \Psi_\pi(p_0, p), \quad (3.11)$$

where the quantity  $\Psi_\pi$  satisfies the equation

$$\Psi_\pi(p_0, p) = \xi_\pi(p_0, p) + \int d^3k \Psi_\pi(p_0, k) G(p_0, k) V_\pi(k, p), \quad (3.12)$$

with

$$\xi_\pi(p_0, p) = \int d^3k \tilde{G}(p_0, k) V_\pi(k, p) + \delta z, \quad (3.13)$$

or symbolically

$$\Psi_\pi = \xi_\pi + \Psi_\pi G V_\pi, \quad \xi_\pi = \xi_1 \tilde{G} V_\pi + \delta z. \quad (3.14)$$

The  $\tilde{G} V_\pi$  term in  $\xi_\pi$  contains logarithmic divergence, which can be removed by adjusting the one-loop counter-term  $\delta z$  to

$$\delta z = - \int d^3k \tilde{G}(m, k) V_\pi(k, 0), \quad (3.15)$$

so that the quantities  $\xi_\pi$ ,  $\Psi_\pi$  and  $\Xi_2$  become finite. Moreover from Eq. (2.5) it follows that  $\xi_\pi \xrightarrow{p \rightarrow \infty} O(\ln p)$ ,  $\Psi_\pi \xrightarrow{p \rightarrow \infty} O(\ln p)$ .

We now proceed with the renormalization of  $\mathcal{X}$ , which in our scheme reduces to a subtractive renormalization of  $\Sigma$ . The term in  $\xi G \xi^T$  (the first diagram in the right-hand side of the third line of Fig. 1), which is  $\delta z$ -independent, contains divergences with energy-dependent coefficients. These divergences can be consistently subtracted using the BPHZ prescription. Those terms in  $\xi G \xi^T$  which contain  $\delta z$  linearly cancel the sub-divergences in the  $\delta z$ -independent part of the two-loop diagram  $\xi G V_\pi G \xi^T$  contained in the  $\xi G T_\pi G \xi^T$  part of the quantity  $\Sigma$  (second diagram in the righthand side of the third line of Fig. 1). The overall divergence of the two-loop diagram  $\xi G V_\pi G \xi^T$  requires an additional BPHZ subtraction. Terms in  $\xi G \xi^T$  which contain  $\delta z$  quadratically cancel the two-loop sub-divergence in the  $\delta z$ -independent part of the three-loop diagram  $\xi G V_\pi G V_\pi G \xi^T$  (the last explicitly shown diagram in the third line of Fig. 1). In addition, all  $\delta z$ -dependent parts of  $\xi G \xi^T$  require an additional subtraction of overall divergences. All other divergences appearing in the loop expansion of  $\xi G T_\pi G \xi^T$  are canceled automatically by contributions of the  $\delta z$  counter term. For example, the one-loop sub-divergences of the  $\delta z$ -independent part of the three loop diagram  $\xi G V_\pi G V_\pi G \xi^T$  are canceled by those expressions generated by the diagram  $\xi G V_\pi G \xi^T$ , which are linear in  $\delta z$ . In the following, we provide the explicit expressions needed to compute the quantity  $\Sigma$  in Eq. (2.13) and define the corresponding subtractions. It is convenient to split  $\Sigma$  into three terms

$$\Sigma = \Sigma_0 + \Sigma_\pi^{\text{finite}} + \Sigma_\pi^{\text{div}}. \quad (3.16)$$

The term  $\Sigma_0$  contains only “pionless” contributions:

$$\Sigma_0 = \xi G \xi^T|_{\delta z=0} \equiv \begin{pmatrix} I_0(q) & I_2(q) \\ I_2(q) & I_4(q) \end{pmatrix}, \quad (3.17)$$

where we have introduced the integrals

$$\{I_0(q), I_2(q), I_4(q)\} = \int d^3k \{1, \vec{k}^2, (\vec{k}^2)^2\} G(p_0, k). \quad (3.18)$$

We subtract the infinite local (polynomial in  $p_0 - m$ ) terms from these integrals to make them finite, so that  $I_i(q)$  ( $i = 0, 2, 4$ ) are replaced with the subtracted  $I_i^R(\mu, q)$  defined as

$$\begin{aligned}
I_0^R(\mu, q) &= I_0(q) - I_0(i\mu) = \frac{m^2}{8\pi^2 p_0} \left[ 2q \left( \sinh^{-1} \frac{q}{m} - i\pi \right) - \pi m \right] \\
&\quad + \frac{m^2}{8\pi^2 \sqrt{m^2 - \mu^2}} \left[ 2\mu \left( \sin^{-1} \frac{\mu}{m} - \pi \right) + \pi m \right], \\
I_2^R(\mu, q) &= I_2(q) - q^2 I_0(i\mu) - \int d^3k \tilde{G}(p_0, k) = q^2 I_0^R(\mu, q), \\
I_4^R(\mu, q) &= I_4(q) - q^4 I_0(i\mu) - \int d^3k (k^2 + q^2) \tilde{G}(p_0, k) = q^4 I_0^R(\mu, q). \tag{3.19}
\end{aligned}$$

The subtracted integrals depend on the choice of the subtraction point  $\mu$ . In principle, one has an additional freedom in fixing finite terms polynomial in  $p_0 - m$ . However, it leads to higher-order effects, therefore we will only study the  $\mu$  dependence of obtained results.

The remaining terms of  $\Sigma$  are split into the finite and divergent parts,  $\Sigma_\pi^{\text{finite}}$  and  $\Sigma_\pi^{\text{div}}$ , which are given by

$$\begin{aligned}
\Sigma_\pi^{\text{finite}} &= \begin{pmatrix} 1 & q^2 \\ q^2 & q^4 \end{pmatrix} \xi_1 G V_\pi G \Xi_1 + \begin{pmatrix} 0 & 1 \\ 1 & 2q^2 \end{pmatrix} \xi_1 G V_\pi G \Psi_\pi \\
&\quad + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \xi_\pi G V_\pi G \Psi_\pi, \tag{3.20}
\end{aligned}$$

and

$$\Sigma_\pi^{\text{div}} = \begin{pmatrix} 0 & 1 \\ 1 & 2q^2 \end{pmatrix} \xi_1 G \xi_\pi + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left\{ \xi_1 \tilde{G} V_\pi \tilde{G} \xi_1 + 2\delta z \xi_1 \tilde{G} \xi_1 + \xi_\pi G \xi_\pi \right\}. \tag{3.21}$$

It is straightforward to show using Eqs. (3.9), (3.11) and (3.14) that two expressions for  $\Sigma$  given by Eq. (2.13) and Eq. (3.16) are identical. Note that the divergent part  $\Sigma_\pi^{\text{div}}$  contains only a finite number of iterations of the OPE potential (up to three loops as shown in Fig. 1). All nonperturbative effects due to OPE are included in  $\Sigma_\pi^{\text{finite}}$ .

Again, following the BPHZ procedure, we subtract the infinite local terms containing overall divergencies from  $\Sigma_\pi^{\text{div}}$  of the following form

$$\delta\Sigma_\pi = \begin{pmatrix} 0 & 1 \\ 1 & 2q^2 \end{pmatrix} \xi_1 G_0 \xi_\pi + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left\{ \xi_1 \tilde{G} V_\pi \tilde{G} \xi_1 + 2\delta z \xi_1 \tilde{G} \xi_1 + \xi_\pi G_0 \xi_\pi \right\}, \tag{3.22}$$

where  $G_0(k) \equiv G(p_0 = m, k)$ , so that the full subtracted result for  $\Sigma$  reads

$$\begin{aligned}
\Sigma^R &= \begin{pmatrix} 1 & q^2 \\ q^2 & q^4 \end{pmatrix} \left\{ I_0^R(\mu, q) + \xi_1 G V_\pi G \Xi_1 \right\} + \begin{pmatrix} 0 & 1 \\ 1 & 2q^2 \end{pmatrix} \left\{ \xi_1 (G - G_0) \xi_\pi + \xi_1 G V_\pi G \Psi_\pi \right\} \\
&\quad + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left\{ \xi_\pi (G - G_0) \xi_\pi + \xi_\pi G V_\pi G \Psi_\pi \right\}. \tag{3.23}
\end{aligned}$$

The finiteness of  $\Sigma^R$  can be shown using ultraviolet behavior of  $V_\pi$ ,  $\Xi_1$ ,  $\xi_\pi$  and  $\Psi_\pi$  considered above. Performing subtractions in the spirit of chiral effective field theory we were supposed to also expand in powers of the pion mass, which we have not done here. However, the



non-analytic dependence of the resulting subtraction terms on the pion mass is of a higher order relative to the accuracy of our calculation. Note also that our final perturbative result which will be discussed in section IV depends on the choice of the renormalization scheme. This dependence is also of higher order.

In final finite expressions for  $\mathcal{X}$  we substitute the finite renormalized couplings for  $\tilde{C}$  and  $C_2^3$ :

$$\mathcal{X}^R = \left[ (C^R)^{-1} - \Sigma^R \right]^{-1}. \quad (3.24)$$

Note that  $\mathcal{X}^R$  is not equal to  $\mathcal{X}$ , because not all of the divergencies can be absorbed by means of redefinition of two available low-energy constants. We parameterize effectively the dependence of the result on the renormalization scheme by exploiting the freedom to choose the subtraction point  $\mu$ .

#### IV. THE CHOICE OF RENORMALIZATION CONDITIONS AND NUMERICAL RESULTS

We are now in the position to specify the choice of renormalization conditions which, in the case at hand, translates into specifying the subtraction point  $\mu$ . Notice that we have already made a specific choice for the subtractions of the integrals  $I_2(q)$  and  $I_4(q)$  in Eq. (3.18). It is useful to recall the key aspects of renormalization in the simple case of pionless EFT corresponding to  $g_A = 0$ , see e.g. Refs. [10, 42], before dealing with the more complicated pionfull approach. To be specific, consider the NN S-wave scattering amplitude corresponding to the contact interaction potential of Eq. (2.10),

$$T_{\text{cont}} = -\frac{2\{C_2^2 m^2 [I(q) q^4 - 2q^2 I_2(q) + I_4(q)] + 4C_2 q^2 + 2C\}}{C_2^2 m^4 [I(q) I_4(q) - I_2(q)^2] + 4C_2 m^2 I_2(q) + 2CI(q)m^2 - 4}. \quad (4.1)$$

To make the following discussion more transparent, we restrict ourselves to the leading nonrelativistic approximation so that the above expression takes the form

$$T_{\text{cont}} = -\frac{C_2^2 m [q^4 J(q) - 2q^2 J_2(q) + J_4(q)] + 2C_2 q^2 + C}{mJ(q) [C_2^2 m J_4(q) + C] - [C_2 m J_2(q) - 1]^2}, \quad (4.2)$$

where

$$\{J(q), J_2(q), J_4(q)\} = \frac{2}{m} \int \frac{d^3 k}{(2\pi)^3} \frac{\{1, \vec{k}^2, (\vec{k}^2)^2\}}{q^2 - k^2 + i\epsilon}. \quad (4.3)$$

We have verified via explicit calculations that the omitted  $1/m$ -corrections are heavily suppressed for the problem at hand and of no relevance for the forthcoming discussion. Using dimensional regularization we express  $J_2(q^2)$  and  $J_4(q^2)$  in terms of  $J(q^2)$  as

$$J_2(q^2) = q^2 J(q^2), \quad J_4(q^2) = q^4 J(q^2) \quad (4.4)$$

and subtract  $J(q^2)$  at  $q^2 = -\mu^2$  obtaining

$$J^R(q^2) = -\frac{i q + \mu}{2\pi m}. \quad (4.5)$$

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<sup>3</sup> It is not possible to disentangle the  $D$  term from the fitted value of  $\tilde{C}^R$ .

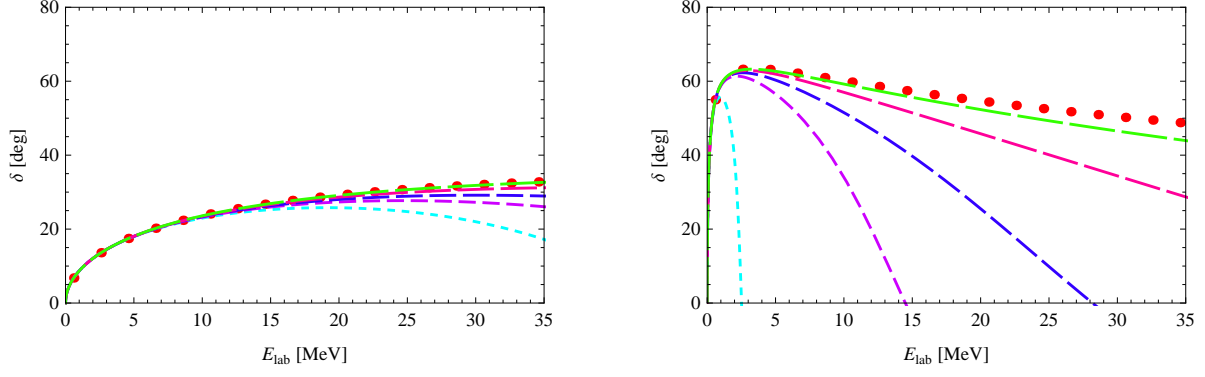


FIG. 2: Renormalization scale dependence of the phase shift in  $^1S_0$  partial wave NN scattering emerging by non-perturbative inclusion of the NLO contact interaction in pionless EFT. Left panel corresponds to a natural scattering length while right panel shows the case of an unnaturally large scattering length. Circles (color online: red) on both panels refer to the synthetic data as described in the text while the dashed curves with increasing dash length correspond to  $\mu = 1, 50, 100, 200, 400$  MeV.

Subtracted integrals  $J_2^R(q^2)$  and  $J_4^R(q^2)$  are obtained by replacing  $J(q^2)$  in Eq. (4.4) by  $J^R(q^2)$  specified in Eq. (4.5). The renormalized amplitude is then given by Eq. (4.2) with the divergent integrals being replaced by their subtracted values and the bare LECs  $C$  and  $C_2$  being replaced by the renormalized ones  $C^R(\mu)$  and  $C_2^R(\mu)$  [42]. Using the scattering length  $a$  and effective range  $r$  to determine these two LECs, we obtain the renormalized expression for the effective range function  $q \cot \delta$  in terms of observable quantities

$$q \cot \delta = \frac{-a^2 r \mu q^2 + 2a\mu - 2}{a(arq^2 - 2a\mu + 2)} = -\frac{1}{a} + \frac{rq^2}{2} + \frac{ar^2q^4}{4(a\mu - 1)} + \frac{a^2r^3q^6}{8(a\mu - 1)^2} + \dots \quad (4.6)$$

Notice that the resulting expression is explicitly  $\mu$ -dependent. This is because the UV divergencies emerging from iterations of the LS equation require counter terms beyond the truncated potential unless the  $C_2$  and higher-order interactions are treated in perturbation theory. For the natural case describing a perturbative scenario corresponding to  $a \sim \Lambda^{-1}$ ,  $r \sim \Lambda^{-1}$ , ..., with  $\Lambda$  being the hard scale of the order of  $\Lambda \sim M_\pi$ , it is appropriate to choose the subtraction scale  $\mu$  of the order of the soft scale in the problem, i.e. of the order of external momenta of the nucleons  $\mu \sim q \ll \Lambda$ . This ensures that the values of the shape parameters  $v_i$  in Eq. (4.6) scale with the corresponding powers of  $\Lambda$ , so that the residual  $\mu$ -dependence in the amplitude is beyond the accuracy of the NLO approximation. This is visualized in the left panel of Fig. 2, where the NLO pionless EFT predictions for the phase shift for the case of  $a = -M_\pi^{-1}$  fm and  $r = M_\pi^{-1}$  fm are shown as a function of laboratory energy  $E_{\text{lab}}$  for different choices of the subtraction scale  $\mu$ . The "data" in Fig. 2 correspond to the effective range approximation with all shape coefficients set to zero. Notice that choosing  $\mu$  of the order of the hard scale also results in a valid low-energy expansion of the effective-range function as visualized in the figure.

On the other hand, for the unnatural case describing the non-perturbative situation of a system being close to the unitary limit and corresponding to very large values of the

scattering length,  $a \rightarrow \infty$ , one obtains from Eq. (4.6) the expansion

$$q \cot \delta = -\frac{r\mu q^2}{rq^2 - 2\mu} = \frac{rq^2}{2} + \frac{r^2q^4}{4\mu} + \frac{r^3q^6}{8\mu^2} + \dots \quad (4.7)$$

As it is clear from Eq. (4.7), the generated (scheme-dependent) coefficients of the effective range expansion will be unnaturally large if one chooses  $\mu$  of the order of the soft scale in the problem. In the KSW approach, one compensates for these large contributions by taking them into account perturbatively and canceling against the contributions of the corresponding higher-order contact interactions which are also assumed to be unnaturally large (i.e. the assumed scaling of the corresponding LECs involves powers of the soft scale). We solve the problem of the unnaturally large scattering length by choosing the subtraction point  $\mu$  of the order of the *hard scale* in the problem,  $\mu \sim \Lambda$ . This guarantees that no large contributions in the induced coefficients of the effective range expansion are generated and the  $\mu$ -dependence of the scattering amplitude is indeed beyond the order one is working at, see Eq. (4.6). This is visualized in the right panel of Fig. 2. Notice that choosing  $\mu \ll \Lambda$  leads to strong distortions in the phase shifts and thus considerably restricts the range of applicability of pionless EFT which is expected to be valid for energies up to  $E_{\text{lab}} = M_\pi^2/(2m) \sim 10.5$  MeV. Before turning to the pionfull EFT we are actually interested in, it is important to emphasize that the subtraction scale  $\mu$  should also not be chosen to be significantly larger than the corresponding hard scale in the problem in order to keep  $\mu$ -dependent terms beyond the accuracy of the calculation. This feature cannot be illustrated in the considered example of pionless EFT, where taking the limit  $\mu \rightarrow \infty$  simply leads to vanishing shape parameters. In the presence of a long-range interaction, the induced  $\mu$ -dependent contributions in Eq. (4.6) will, in general, involve the mass scale associated with the long-range interaction and positive powers of  $\mu$ . Choosing  $\mu \gg \Lambda$  will then enhance the scheme-dependent contributions, which are nominally of a higher order, and spoil the predictive power of a theory. An explicit example of such a “peratization” is considered in Ref. [10].

After these introductory remarks, we are now in the position to present our results for the  $^1S_0$  phase shift at NLO in chiral EFT. We employ the exact isospin symmetry as appropriate at LO and use the following values for the LECs entering the OPE potential

$$M_\pi = 138 \text{ MeV}, \quad F_\pi = 92.4 \text{ MeV}, \quad g_A = 1.267. \quad (4.8)$$

The numerical value of the renormalized LECs  $\tilde{C}^R(\mu)$  and  $C_2^R(\mu)$  are determined from a fit to the neutron-proton  $^1S_0$  phase shift of the Nijmegen partial-wave analysis (PWA) [43] in the energy range of  $0 \dots 50$  MeV for several choices of the subtraction point  $\mu$  as discussed above. The resulting phase shifts for different choices of  $\mu$  are plotted in Fig. 3. As expected and explained at the beginning of this section, we do observe some residual  $\mu$ -dependence of the predicted phase shifts which gets strongly enhanced if one chooses  $\mu$  of the order or smaller than  $M_\pi$ . On the other hand, for the appropriate choice of the subtraction scale  $\mu \sim \Lambda$ , where the hard scale  $\Lambda$  can be realistically estimated to be of the order of  $\Lambda \sim 600\text{--}700$  MeV<sup>4</sup>, the dependence on  $\mu$  appears to be moderate, and the predicted energy dependence

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<sup>4</sup> These values are of the order of the masses of the sigma and rho mesons which phenomenologically are known to yield the most important short-range contribution to the nucleon-nucleon potential [44]. This estimation also agrees well with the findings of chiral EFT calculations utilizing a finite cutoff [38, 39]. A deeper discussion on the breakdown scale of nuclear chiral EFT can be found in Ref. [17].

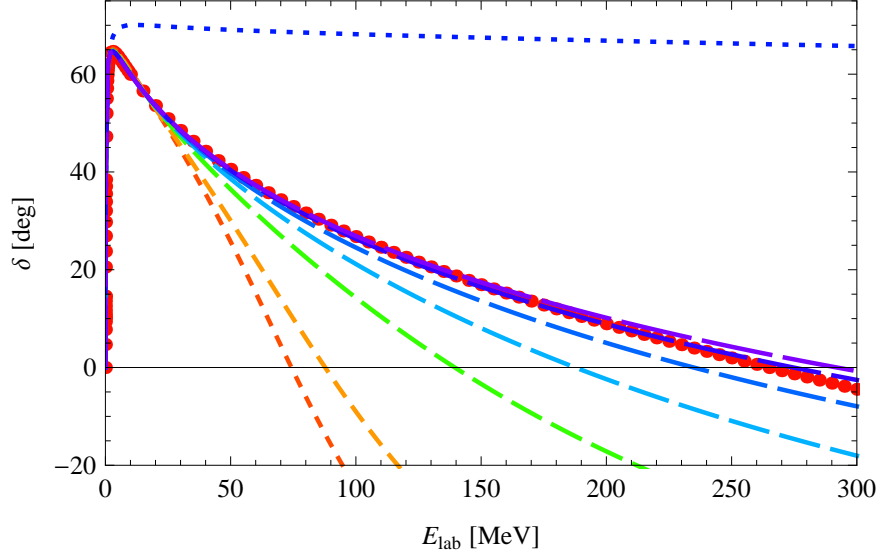


FIG. 3: Neutron-proton  $^1S_0$  phase shifts versus the energy in the laboratory frame. Circles (red) correspond to the Nijmegen PWA [43]. Dotted line represents the LO result. Curves with different dashed (colors) correspond to non-perturbative inclusion of the NLO contact interaction potential for  $\mu = 50, 100, 300, 500, 700, 850$  and  $900$  MeV respectively.

of the phase shift shows a good agreement with the Nijmegen PWA. Choosing  $\mu \sim \Lambda$ , the observed  $\mu$ -dependence of the phase shift is considerably smaller than the difference between the LO and NLO results and can serve as an estimation of the size of corrections beyond NLO, i.e. it defines the lower bound for the theoretical uncertainty of our calculation. It is especially comforting to see that the spread of predictions for the subtraction point chosen in the range of  $\mu = 500 \dots 900$  MeV matches very well the estimated theoretical accuracy at NLO in calculations based on a finite cutoff, see Fig. 9 of Ref. [38]

It is also interesting to address the question of perturbativeness of the subleading short-range interaction within our scheme. Given that calculating phase shifts always relies on some kind of unitarization procedure, it is more appropriate to address this issue by looking at the scattering amplitude directly. To be specific, consider the ratio  $R(E_{\text{lab}})$  defined as

$$R(E_{\text{lab}}) = \frac{|T^{\text{NLO}}(E_{\text{lab}})|}{|T^{\text{LO}}(E_{\text{lab}})|}, \quad (4.9)$$

where  $T^{\text{LO}}$  and  $T^{\text{NLO}}$  denote the  $T$ -matrix calculated at LO and up to NLO, respectively. In Fig. 4 we show by the solid line the quantity  $R$  based on the nonperturbative inclusion of the subleading contact interaction as described above and corresponding to the choice of  $\mu = 850$  MeV.<sup>5</sup> Notice that the subleading contribution to the amplitude becomes comparable in size with the leading one at higher energies, the feature that could have been expected by looking at the LO prediction for the  $^1S_0$  phase shift. We also plot in this figure the ratio  $R$  resulting from the inclusion of the subleading contact interaction in first-order perturbation

<sup>5</sup> Note that the ratio  $R$  is not equal to 1 at threshold, because in our subtraction scheme diagrams containing NLO contact interactions but no overall divergencies are not subtracted.

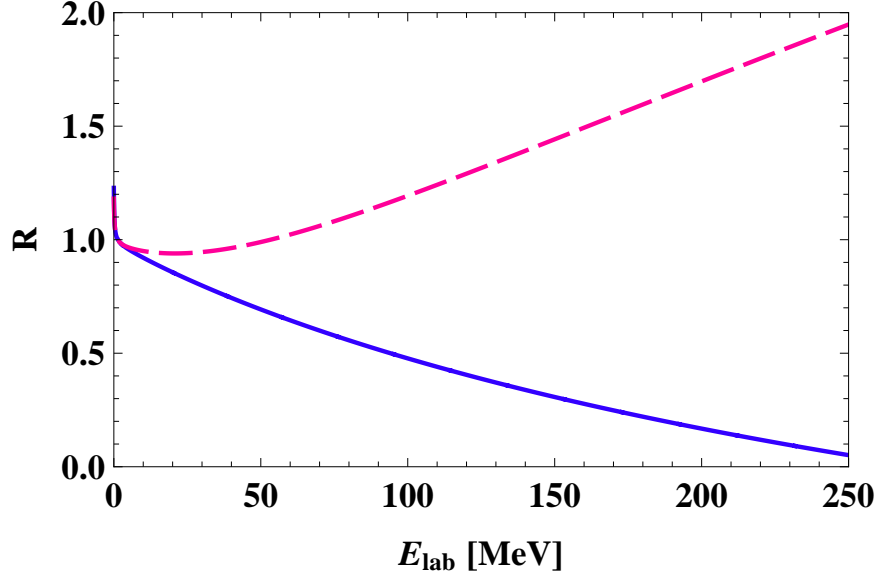


FIG. 4: The ratio  $R$  defined in Eq. (4.9) versus the energy in the laboratory frame. The solid (dashed) line shows the result based on  $T^{\text{NLO}}$  corresponding to the nonperturbative (perturbative) inclusion of the subleading contact interaction. In both cases, the subtraction scale is set to  $\mu = 850$  MeV.

TABLE I: Predictions for the coefficients in the effective range expansion of the  $^1S_0$  phase shifts (low-energy theorems) at LO and NLO in the modified Weinberg approach in comparison with the NLO KSW predictions of Ref. [14] and empirical numbers extracted from the Nijmegen PWA [45, 46]. For the NLO Weinberg results, we show the predictions corresponding to the variation of the subtraction point in the range of  $\mu = 500 \dots 900$  MeV. The errors quoted for the LO predictions refer to the uncertainty in the numerical extraction of the coefficients [1].

	$a$ [fm]	$r$ [fm]	$v_2$ [fm <sup>3</sup> ]	$v_3$ [fm <sup>5</sup> ]	$v_4$ [fm <sup>7</sup> ]
LO, Ref. [1]	fit	1.50	-1.9	8.6(8)	-37(10)
NLO, nonperturb. $C_2$	fit	fit	-0.55 ... -0.61	5.1 ... 5.5	-29.6 ... -30.8
NLO, perturbative $C_2$	fit	fit	-0.51 ... -0.57	4.5 ... 4.7	-28.8 ... -29.8
NLO KSW, Ref. [14]	fit	fit	-3.3	18	-108
Nijmegen PWA	-23.7	2.67	-0.5	4.0	-20

theory for the same choice of  $\mu$  (and using the same values for the renormalized low-energy constants as determined in the nonperturbative calculation). This shows clearly that it is advantageous to include the subleading contact interaction nonperturbatively within the employed framework for energies of about  $E_{\text{lab}} \sim 50$  MeV and higher.

Last but not least, we also give the predictions for the coefficients in the effective range expansion which may be regarded as low-energy theorems (LETs), see [10, 14] for more details. In table I, the LETs in the KSW and Weinberg approaches are confronted with the results of the Nijmegen PWA for the  $^1S_0$  partial wave, respectively. We observe a clear improvement in the reproduction of the LETs when going from LO to NLO. Notice that

the nonperturbative treatment of the subleading contact interaction appears to have minor effect for the LETs. It should, however, be emphasized that the extraction of the coefficients in the effective range expansion requires performing a unitarization of the amplitude which provides a partial resummation of  $C_2$ -contributions.

## V. SUMMARY AND CONCLUSIONS

In this paper we have considered nucleon-nucleon scattering in the  $^1S_0$  partial wave within the modified Weinberg approach. The integral equation based on the leading-order potential, which consists of the momentum- and energy-independent contact interaction and the OPE potential, is renormalizable and was studied in Ref. [1]. The observed large discrepancy between the LO EFT results and the  $^1S_0$  phase shift of the Nijmegen PWA, which starts already at rather low energies, indicates that at least some parts of the higher-order contributions to the effective potential need to be included nonperturbatively. Here we assumed that only the short range part of the NLO potential (in standard Weinberg power counting) needs to be treated non-perturbatively. It involves only the contact interaction terms quadratic in momenta and the pion mass. This makes it possible to perform the subtractive renormalization explicitly in non-perturbative expressions.

The pertinent results of our study can be summarized as follows:

- We have carried out subtractive renormalization of the scattering amplitude based on the potential involving OPE as well as the leading and subleading contact interactions using the framework of Ref. [1] and without relying on perturbation theory.
- The resulting renormalized integral equations for the scattering amplitude have been solved numerically and the values of the renormalized low-energy constants  $\tilde{C}^R(\mu)$  and  $C_2^R(\mu)$  were determined from a fit to phase shifts of the Nijmegen PWA for different choices of the subtraction point  $\mu$ .
- We discussed the issue of the proper choice of renormalization conditions in our scheme and have argued that the observed large value of the scattering length requires choosing the scale  $\mu$ , which corresponds to the renormalization of the LO contact interaction, of the order of the *hard* scale in the problem.
- The resulting predictions for the energy dependence of the  $^1S_0$  phase shift are in a good agreement with the Nijmegen PWA. Moreover, the observed dependence of the phase shifts on the subtraction point  $\mu$  chosen in the range of  $\mu = 500 \dots 900$  MeV agrees well with the theoretical accuracy at NLO estimated in calculations of Refs. [38, 39] based on the standard non-relativistic framework with a finite cutoff.
- We have also addressed perturbativeness of the subleading contact interaction within our scheme. We found that it is advantageous to treat the subleading contact interaction nonperturbatively at energies of about  $E_{\text{lab}} \sim 50$  MeV and higher.
- Finally, we have looked at the low-energy theorems for the coefficients in the effective range expansion and found a clear improvement when going from LO to NLO.

The results of our work open the way to perform higher-order calculations within the modified Weinberg approach proposed in Ref. [1]. As a next step, the role of the two-pion

exchange potential needs to be investigated and the extension to other partial waves has to be performed. Work along these lines is in progress.

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